J Stat Phys (2008) 132: 535–550
DOI 10.1007/s10955-008-9546-7

Global Existence of Classical Solutions to the Fokker-Planck-BGK Equation

Xianwen Zhang

Received: 15 December 2007 / Accepted: 7 April 2008 / Published online: 19 April 2008

Abstract A kinetic model of the Fokker-Planck-Boltzmann equation is introduced by replacing the original Boltzmann collision operator with the Bhatnagar-Gross-Krook collision model (BGK collision model). This model equation, which we call the Fokker-Planck-BGK equation, has many physical features that the Fokker-Planck-Boltzmann equation possesses. We first establish an L^∞ existence result for this equation, by which we construct the approximate solutions. Then, by means of the regularizing effects of the linear Fokker-Planck operator and L^p estimates of local Maxwellians, we obtain some uniform estimates of the approximate solutions. Finally, combining those estimates and regularizing effects, we prove by a compactness argument that the equation has a global classical solution under rather general initial conditions.

Keywords Fokker-Planck-BGK equation · Initial value problem · Classical solution · Local Maxwellian

1 Introduction

In their classical paper [15], R.J. Diperna and P.L. Lions discussed the Cauchy problem of the Fokker-Planck-Boltzmann equation, i.e., the Boltzmann equation perturbed by a Fokker-Planck operator:

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f - \sigma \Delta_\xi f = Q(f, f), \qquad f(0, x, \xi) = f_0(x, \xi),$$

where $Q(f, f)$ is the Boltzmann collision operator, f_0 is an initial datum, and $\sigma > 0$ is the diffusive coefficient. By certain regularizing effects of the Fokker-Planck term $-\sigma \Delta_\xi f$,

Supported by the Scientific Research Foundation of Huazhong University of Science and Technology (HUST-SRF).

X. Zhang (✉)
Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, People's Republic of China
e-mail: xwzhang@mail.hust.edu.cn

they established a global existence result of renormalized solutions to this equation. Their basic idea to define renormalized solutions is to regularize the collision term by a quasi-linearizing method [15–17]. Due to the lack of a priori estimates which are strong enough to define the original Boltzmann collision operator, no further regularity results for this kind of solutions have been obtained so far; for example, the question whether this kind of solutions are the ones in the distributional sense is still open.

The above observation suggests that some simplified kinetic models for this equation should be constructed, which can still characterize the evolution of a gas accurately on one hand, and be mathematically manageable on the other hand. It is well known that, in rarefied gas dynamics, there is a good approximation of the Boltzmann collision integral, i.e., the Bhatnagar-Gross-Krook collision model (abbreviated to "BGK model"). This model was introduced in [6] and is simpler than the Boltzmann collision operator (for a complete discussion of the BGK model, we refer the readers to [13]). In this paper, by replacing the Boltzmann collision operator in the above equation with the BGK collision model, we obtain a kind of kinetic model which we call the Fokker-Planck-BGK equation. As we shall see later, this simplified kinetic model can be handled more easily than the Fokker-Planck-Boltzmann equation; for example, we will show the global existence of classical solutions under rather general conditions.

For a given rarefied gas, let $f : (t, x, \xi) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty]$ be its microscopic kinetic density of particles at time t and position x, moving with velocity ξ, its macroscopic quantities (i.e., hydrodynamical variables) are defined by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2 + N\rho\theta \end{pmatrix}(t, x) = \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ \xi \\ |\xi|^2 \end{pmatrix} f(t, x, \xi) d\xi, \tag{1.1}$$

where $\rho(t, x)$, $u(t, x)$, and $\theta(t, x)$ are, respectively, the mass density, bulk velocity, and temperature of the gas at time t and position x. Then the Fokker-Planck-BGK equation in $\mathbb{R}^N$ ($N \geq 1$), as described above, is the following nonlinear evolutional equation if an initial distributional function $f_0(x, \xi) \geq 0$ is prescribed

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f - \sigma \Delta_\xi f = M[f] - f, \qquad f(0, x, \xi) = f_0(x, \xi), \tag{1.2}$$

where $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, $t \geq 0$, and $J(f) = M[f] - f$ is the so called BGK collision model in which $M[f]$ is the local Maxwellian

$$M[f](t, x, \xi) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{N/2}} \exp\left(-\frac{|\xi - u(t, x)|^2}{2\theta(t, x)}\right). \tag{1.3}$$

This kinetic equation shares many characteristic features of the Fokker-Planck-Boltzmann equation, for example, the conservation of mass and momentum, and linear increase of kinetic energy; these facts are easily verified at least at a formal level and will be strictly proved in this paper. Furthermore, the structure of its collision model $J(f)$ is simpler than the Boltzmann operator in the sense that it reduces the complicated integral of interactions to an operator which is computational treatable. On the other hand, the BGK collision model is still very complicated in the mathematical sense due to the occurrence of exponential nonlinearity in (1.3).

In (1.2), the linear partial differential operator $\sigma \Delta_\xi f$, where $\sigma > 0$ is a constant, is the so-called Fokker-Planck operator. It describes the fact that the gas molecules interact with background medium and their pathes between two interactions obey the Brownian motion, while the BGK collision term $J(f)=M[f]-f$ models two-body interactions in the gas.

On one hand, although the operator $-\xi\cdot\nabla_x f+\sigma\Delta_\xi f$ is degenerately elliptic, it still has certain strong regularizing effects which were previously used by many authors, such as [15] for Fokker-Planck-Boltzmann equation and [7–12, 23, 24] for the Vlasov-Poisson-Fokker-Planck system. In Sect. 2, we will recall and extend those facts so that they can be used in the present paper. On the other hand, a weighted L^p estimate for the BGK collision term $J(f)$ was recently established in [27], this estimate describes that $J(f)$ as a nonlinear operator from a weighted L^p space into itself is bounded. In this paper, we will make fully use of the above two points to construct global classical solutions to the Fokker-Planck-BGK equation (1.1)–(1.3). Denote by $\rho(t)$, $M(t)$ and $E(t)$ the total mass, momentum and kinetic energy of the gas at time $t\geq 0$, i.e.,

$$(\rho,\ M,\ E)(t)=\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,\xi)\left(1,\xi,|\xi|^2\right)dxd\xi,\quad t\geq 0.$$

The main result of this paper can be described as the following theorem.

Theorem 1.1 *Let $\beta>2$ and $1<p\leq\infty$, and let the initial datum $f_0(x,\xi)$ be a nonnegative function such that*

$$(1+|\xi|^\beta)f_0\in L^1(\mathbb{R}^N\times\mathbb{R}^N),\quad f_0\in L^p(\mathbb{R}^N\times\mathbb{R}^N),\tag{1.4}$$

then there exists a global positive solution $f(t,x,\xi)$ to the Fokker-Planck-BGK equation (1.1)–(1.3) such that

$$f(t,x,\xi)\in C([0,\infty),L^1(\mathbb{R}^N\times\mathbb{R}^N))\cap C^\infty((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$$

and

$$\begin{aligned}&\rho(t)=\rho(0),\quad M(t)=M(0),\quad t\geq 0,\\ &E(t)=E(0)+2N\sigma\rho(0)t,\quad t\geq 0.\end{aligned}\tag{1.5}$$

Furthermore, for any $T<\infty$ there exist positive constants $M_1=M_1(T,\beta,f_0)$ and $M_2=M_2(T,p,f_0)$ such that

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^\beta f(t,x,\xi)dxd\xi\leq M_1,\tag{1.6}$$

$$\sup_{0\leq t\leq T}\|f(t,\cdot)\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}\leq M_2.\tag{1.7}$$

Remark 1.1 Compared with the conditions on the initial data in [15], we do not assume that $f_0(x,\xi)$ has finite entropy and inertia. Here, we only require that the L^p norm of $f_0(x,\xi)$ is finite for some $p>1$. As we have seen, this L^p norm is also propagated by the solutions constructed in Theorem 1.1. On the other hand, an extra assumption on the higher order velocity moment is added to the initial data. We could not weaken this condition to the case $\beta=2$, since the approximate solutions constructed in Sect. 3 can not be proved to produce higher order moments by the present method, unlike the classical BGK equation considered in [21] where it was shown that the free transport equation creates higher order moments.

Before we proceed further, we recall some of the existing results on the classical BGK equation (the Fokker-Planck term disappears in (1.2), i.e., $\sigma = 0$). In 1989, assuming that the total mass, inertia, kinetic energy, and entropy of the initial distribution are finite, i.e., $\int_{\mathbb{R}^N\times\mathbb{R}^N}(1+|x|^2+|\xi|^2+|\log f_0|)f_0(x,\xi)dxd\xi < \infty$, Perthame [21] built an L^1 theory and proved that the classical BGK equation has a positive solution $f(t,x,\xi)\in C([0,\infty),L^1(\mathbb{R}^N\times\mathbb{R}^N))$ in the distributional sense. Moreover this solution conserves mass, momentum, and energy, and satisfies the local entropy identity. Those solutions were proved to propagate the initial higher order moments in [26]. In order to treat uniqueness problem, Perthame and Pulvirenti [22] also developed an L^∞ method and showed that, for x in a periodic domain, there exists a unique polynomially decaying solution if the initial density f_0 does not vanish on any characteristic line; later this result was generalized to the Cauchy problem in [20]. The main ingredient of this method is the estimates on the L^∞ moments of the macroscopic quantities and the local Maxwellians obtained in [22]. It worth mentioning that the initial boundary value problem and its large time behavior were discussed in [1, 14, 19], smooth solutions for small initial data and perturbation of a global Maxwellian were also considered in [2, 3]. Furthermore, some aspects of the hydrodynamical limits were investigated in [4, 5]. Lately, by establishing weighted L^p estimates of the hydrodynamical quantities and local Maxwellians, an L^p existence theorem and certain regularity results were developed in [27] by means of weakly compact argument. As far as we know, for the Fokker-Planck-BGK equation introduced in this paper, Theorem 1.1 is new.

The paper is organized as follows. In Sect. 2, we recall some of the regularizing properties of the linear part in (1.2), which are well-known and frequently used in this field. We mainly emphasize some aspects of those results which are suitable for our present paper. Section 3 is devoted to the construction of approximate solutions and their a priori estimates. Finally, in Sect. 4, we will finish the proof of Theorem 1.1. In this paper, positive constants are denoted by $C(\kappa_1,\kappa,\dots,\kappa_l)$ or $C_{\kappa_1,\kappa,\dots,\kappa_l}$, which depend only on the parameters $\kappa_1,\kappa,\dots,\kappa_l$.

2 Regularity Properties of the Linear Problem

Let us denote by $\mathcal{L}f$ the linear partial differential operator $\xi\cdot\nabla_x f-\sigma\Delta_\xi f$ and consider the following linear evolution equation

$$\frac{\partial}{\partial t}f+\mathcal{L}f=g(t,x,\xi),\qquad f(0,x,\xi)=f_0(x,\xi),\tag{2.1}$$

where f_0 and g are assumed to be known. It is well-known that there exists a fundamental solution $G(t,x,\xi;y,\eta)$ of the linear operator $\frac{\partial}{\partial t}+\mathcal{L}$, which can be expressed by

$$G(t,x,\xi;y,\eta)=G_0(t,x-y-t\eta,\xi-\eta),\tag{2.2}$$

where $x,y,\xi,\eta\in\mathbb{R}^N,\ t\geq 0$, and

$$G_0(t,x,\xi)=\frac{(3/4)^{N/2}}{(\pi\sigma)^N t^{2N}}\exp\left(-\frac{3|x-\frac{t}{2}\xi|^2}{\sigma t^3}\right)\exp\left(-\frac{|\xi|^2}{4\sigma t}\right).$$

Obviously, we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}G_0(t,x,\xi)dxd\xi=1.$$

With the fundamental solution $G(t,x,\xi;y,\eta)$, the solution of (2.1) can be represented by

$$f(t,x,\xi)=\int_{\mathbb{R}^N\times\mathbb{R}^N}G(t,x,\xi;y,\eta)f_0(y,\eta)dyd\eta + \int_0^t ds\int_{\mathbb{R}^N\times\mathbb{R}^N}G(t-s,x,\xi;y,\eta)g(s,y,\eta)dyd\eta. \tag{2.3}$$

For $t\geq 0$, define a linear operator $G(t)$ by

$$G(t)f(x,\xi)=\int_{\mathbb{R}^N\times\mathbb{R}^N}G(t,x,\xi;y,\eta)f(y,\eta)dyd\eta,$$

where f belongs to an L^r space ($1\leq r\leq\infty$). Then, (2.3) is equivalent to

$$f(t,x,\xi)=G(t)f_0(x,\xi)+\int_0^t G(t-s)g(s,x,\xi)ds.$$

The fundamental solution $G(t,x,\xi;y,\eta)$ has many important properties (see, for example, [7–12, 15, 23, 24]), some of which are included in the following two lemmas.

Lemma 2.1 *The following properties hold for the fundamental solution* $G(t,x,\xi;y,\eta)$:

1. *For any* $x,\xi,y,\eta\in\mathbb{R}^N$ *and* $t\geq 0$

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}G(t,z,\zeta;y,\eta)dzd\zeta=1,\qquad \int_{\mathbb{R}^N\times\mathbb{R}^N}G(t,x,\xi;z,\zeta)dzd\zeta=1.$$

2. *For any positive integers* n,k, *any* $2N$ *dimensional multi-indices* α,γ, *and any* $0<\varepsilon<T<\infty$, *there exists a positive constant* $C=C(n,k,\alpha,\gamma,\varepsilon,T)$ *such that for* $t\in[\varepsilon,\ T]$, $x,\xi,y,\eta\in\mathbb{R}^N$

$$\frac{(1+|y|+|\eta|)^n}{(1+|x|+|\xi|)^{n+2(|\alpha|+|\gamma|+2k)}}\left|\partial_t^k\partial_{x,\xi}^{\alpha}\partial_{y,\eta}^{\gamma}G(t,x,\xi;y,\eta)\right|\leq C.$$

3. *For any* $t\geq 0$, *the linear operator* $G(t)$ *satisfies*

$$\|G(t)f\|_{L^p_{x,\xi}}\leq\|f\|_{L^p_{x,\xi}},$$

where $1\leq p\leq\infty$ *and* $1/p+1/p'=1$. *Furthermore,* $\|G(t)f\|_{L^1_{x,\xi}}=\|f\|_{L^1_{x,\xi}}$ *for* $f\geq 0$.

The following lemma reveals the hypo-ellipticity of the linear partial differential operator $\frac{\partial}{\partial t}+\mathcal{L}$, which is proved in [15].

Lemma 2.2 *For* $n=1,2,\ldots$, *let* $g_n\in L^1([0,\ T]\times\mathbb{R}^N\times\mathbb{R}^N)$ *and* $f_0^n\in L^1(\mathbb{R}^N\times\mathbb{R}^N)$ *such that*

$$\lim_{R\to\infty}\sup_{n\geq 1}\int_0^T dt\int_{|x|^2+|\xi|^2>R^2}|g_n(t,x,\xi)|dxd\xi=0,$$

and

$$\lim_{R\to\infty}\sup_{n\geq 1}\int_{|x|^2+|\xi|^2>R^2}|f_0^n(x,\xi)|dxd\xi=0.$$

Suppose that $f_n(t,x,\xi)$ $(n=1,2,\dots)$ *are solutions to*

$$\frac{\partial}{\partial t} f_n + \mathcal{L} f_n = g_n(t,x,\xi), \qquad f_n(0,x,\xi) = f_0^n(x,\xi). \tag{2.4}$$

Then $\{f_n : n=1,2,\dots\}$ *is relatively compact in* $L^1([0,\ T]\times\mathbb{R}^N\times\mathbb{R}^N)$.

For $q, p > 0$ and measurable function $f(x,\xi)$, following [20, 22], we define the norms $\mathbb{N}_q(f)$ and $\mathbb{N}_{q,p}(f)$ as follows

$$\mathbb{N}_q(f) = \sup_{(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} (1+|\xi|^q)|f(x,\xi)|,$$

$$\mathbb{N}_{q,p}(f) = \sup_{(x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} (1+|x|^p)(1+|\xi|^q)|f(x,\xi)|.$$

Then, we have

Lemma 2.3 *There exist two positive constants* $C(N,q)$ *and* $C(N,q,p)$ *such that for any* $t>0$ *and any measurable function* $f(x,\xi)$,

$$\begin{aligned}
\mathbb{N}_q(G(t)f) &\le C(N,q)(1+t^{q/2})\mathbb{N}_q(f),\\
\mathbb{N}_{q,p}(G(t)f) &\le C(N,q,p)\max\{1,\ t^p\}(1+t^{\frac{q}{2}+2p})\\
&\quad \times [\mathbb{N}_{q,p}(f) + t^p\mathbb{N}_{q+p}(f)].
\end{aligned}$$

Proof Denote $(1+|x|^p)(1+|\xi|^q)|G(t)f(x,\xi)|$ by $I(t,x,\xi)$. Then, by the Peetre's inequality and the definitions of the norms $\mathbb{N}_q$ and $\mathbb{N}_{q,p}$, we have

$$\begin{aligned}
I(t,x,\xi) &\le \int_{\mathbb{R}^N\times\mathbb{R}^N} G(t,x,\xi;y,\eta)(1+|x|^p)(1+|\xi|^q)|f(y,\eta)|dyd\eta\\
&\le C_{q,p}\int_{\mathbb{R}^N\times\mathbb{R}^N} G_0(t,x-y-t\eta,\xi-\eta)(1+|\xi-\eta|^q)(1+|\eta|^q)\\
&\quad \times (1+|x-y-t\eta|^p)(1+|y+t\eta|^p)|f(y,\eta)|dyd\eta\\
&\le C_{q,p}[\mathbb{N}_{q,p}(f)+t^p\mathbb{N}_{q+p}(f)]\int_{\mathbb{R}^N\times\mathbb{R}^N} G_0(t,x-y-t\eta,\xi-\eta)\\
&\quad \times (1+|x-y-t\eta|^p)(1+|\xi-\eta|^q)dyd\eta.
\end{aligned}$$

Then, we get by setting $z=x-y-t\eta,\ \zeta=\xi-\eta$

$$\begin{aligned}
I(t,x,\xi) &\le C_{q,p}[\mathbb{N}_{q,p}(f)+t^p\mathbb{N}_{q+p}(f)]\int_{\mathbb{R}^N\times\mathbb{R}^N} G_0(t,z,\zeta)\\
&\quad \times (1+|z|^p)(1+|\zeta|^q)dzd\zeta\\
&\le C_{q,p}\max\{1,\ t^p\}[\mathbb{N}_{q,p}(f)+t^p\mathbb{N}_{q+p}(f)]\int_{\mathbb{R}^N\times\mathbb{R}^N} G_0(t,z,\zeta)\\
&\quad \times \left(1+\left|z-\frac{t}{2}\zeta\right|^p\right)\left(1+|\zeta|^{q+p}\right)dzd\zeta.
\end{aligned}$$

On the other hand, by the definition of $G_0(t,z,\zeta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} G_0(t,z,\zeta)\left(1+\left|z-\frac{t}{2}\zeta\right|^p\right)\left(1+|\zeta|^{q+p}\right)dzd\zeta\\
&=\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(1+\left|z-\frac{t}{2}\zeta\right|^p\right)\left(1+|\zeta|^{q+p}\right)\\
&\quad\times\frac{(3/4)^{N/2}}{(\pi\sigma)^N t^{2N}}\exp\left(-\frac{3|z-\frac{t}{2}\zeta|^2}{\sigma t^3}\right)\exp\left(-\frac{|\zeta|^2}{4\sigma t}\right)dzd\zeta\\
&=\int_{\mathbb{R}^N}(1+|\zeta|^{q+p})\left(\frac{1}{4\pi\sigma t}\right)^{N/2}\exp\left(-\frac{|\zeta|^2}{4\sigma t}\right)d\zeta\\
&\quad\times\int_{\mathbb{R}^N}(1+|z|^p)\left(\frac{3}{\pi\sigma t^3}\right)^{N/2}\exp\left(-\frac{3|z|^2}{\sigma t^3}\right)dz\\
&=C_1(N,q,p)(1+t^{\frac{q}{2}+2p})<\infty.
\end{aligned}$$

Inserting this inequality into the previous one, we obtain

$$I(t,x,\xi)\le C(N,q,p)\max\{1,\ t^p\}(1+t^{\frac{q}{2}+2p})[\mathbb{N}_{q,p}(f)+t^p\mathbb{N}_{q+p}(f)],$$

where $C(N,q,p)=C_{q,p}C_1(N,q,p)$. This implies the second inequality. The proof of the first inequality is similar to that of the second one and the proof is complete. □

In order to construct appropriate approximate solutions, we need another result which was firstly shown by Mischler [20] for the classical BGK equation.

Lemma 2.4 *Suppose that there exist positive constants* ϵ, r *and a nonnegative function* $\phi(\xi)\in L^1(\mathbb{R}^N)$ *such that* $\phi(\xi)\ge\epsilon$ *for* $|\xi|<1$ *and*

$$f_0(x,\xi)\ge\frac{\phi(\xi)}{1+|x|^r},\quad \textit{for } (x,\xi)\in\mathbb{R}^N\times\mathbb{R}^N\ a.e. \tag{2.5}$$

Let f *be the solution to* (2.1) *such that* $g\ge -f$, *then there exists a positive constant* δ *such that*

$$\rho(t,x)=\int_{\mathbb{R}^N} f(t,x,\xi)d\xi\ge\frac{\delta}{1+|x|^r},\quad \textit{for } (t,x)\in[0,\ T]\times\mathbb{R}^N\ a.e.$$

Proof By rewriting (2.1) as $(\frac{\partial}{\partial t}+\mathcal{L})(\exp(t)f)=\exp(t)(g+f)$, we obtain

$$f(t,x,\xi)=G(t)f_0(x,\xi)+\int_0^t\exp(s-t)G(t-s)[g(s,x,\xi)+f(s,x,\xi)]ds,$$

which implies by the assumptions that

$$\begin{aligned}
f(t,x,\xi)\ge G(t)f_0(x,\xi)&=\int_{\mathbb{R}^N\times\mathbb{R}^N}G(t,x,\xi;y,\eta)f_0(y,\eta)dyd\eta\\
&\ge\int_{\mathbb{R}^N\times\{|\eta|<1\}}G(t,x,\xi;y,\eta)\frac{\phi(\eta)}{1+|y|^r}dyd\eta
\end{aligned}$$

$$\geq \epsilon \int_{\mathbb{R}^N \times \{|\eta|<1\}} G_0(t, x-y-t\eta, \xi-\eta) \frac{1}{1+|y|^r} dy d\eta.$$

It follows that

$$\begin{aligned} \rho(t,x) &\geq \epsilon \int_{\mathbb{R}^N} d\xi \int_{\mathbb{R}^N \times \{|\eta|<1\}} \frac{G_0(t,y,\xi)}{1+|x-y-t\eta|^r} dy d\eta \\ &\geq \epsilon C_r(1+T^r)\omega_N \int_{\mathbb{R}^N} d\xi \int_{\mathbb{R}^N} \frac{G_0(t,y,\xi)(1+|y|^r)}{1+|x|^r} dy, \end{aligned}$$

where ω_N is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^N$. Since

$$\int_{\mathbb{R}^N} d\xi \int_{\mathbb{R}^N} G_0(t,y,\xi)(1+|y|^r) dy \leq C(T,r) < \infty,$$

as shown in the proof of Lemma 2.3, we obtain that

$$\rho(t,x) \geq \frac{\delta}{1+|x|^r},$$

where $\delta = \delta(N,T,r,\epsilon) > 0$. The proof is complete. □

3 Estimates of Approximate Solutions

In this section, we construct approximate solutions to (1.1)–(1.3), and then we prove some a priori estimates of these solutions. In order to do so, we first recall the L^p estimates of the local Maxwellian $M[f]$ established in [20, 22, 27].

Lemma 3.1 *Let $f(x,\xi)$ be any positive measurable function. Suppose $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then for $q \in \{0\} \cup (1,\ \frac{p-1}{p}N) \cup (\frac{p-1}{p}N+2,\ \infty)$,*

$$\| |\xi|^q M[f] \|_{L^p} \leq C(N,q,p) \| |\xi|^q f \|_{L^p}. \tag{3.1}$$

Furthermore, for $p = \infty$, $q \in (N+2,\ \infty)$, and any $r \geq 0$, we have

$$\mathbb{N}_q(M[f]) \leq C(N,q)\mathbb{N}_q(f), \tag{3.2}$$

$$\mathbb{N}_{q,r}(M[f]) \leq C(N,q,r)\mathbb{N}_{q,r}(f). \tag{3.3}$$

Then, with the idea introduced by Perthame and Pulvirenti [22], and Mischler [20], we establish an existence and uniqueness result for L^∞ initial data, which is the starting point of our method and is meaningful by itself as well.

Proposition 3.2 *Suppose that the initial datum $f_0(x,\xi)$ satisfies the lower bound condition (2.5) for $r > N$, and suppose also that $\mathbb{N}_{q+r}(f_0) < \infty$ and $\mathbb{N}_{q,r}(f_0) < \infty$ for some $q > N+2$. Then there exists a unique positive solution $f(t,x,\xi)$ to (1.1)–(1.3) satisfying that for any $0 < T < \infty$ there exists a positive constant $C(T)$ such that $\mathbb{N}_{q,r}(f(t)) \leq C(T)$, $\mathbb{N}_{q+r}(f(t)) \leq C(T)$ for $t \in [0,\ T]$.*

Proof Given $T \in (0, \infty)$, let $f(t,x,\xi)$ be a positive solution to (1.1)–(1.3) for $t \in [0, T]$, then by Lemmas 2.3 and 3.1–3.3, we have that for $t \in [0, T]$,

$$\begin{aligned}\mathbb{N}_{q+r}(f(t)) &\le C(N,q+r)(1+t^{(q+r)/2})\mathbb{N}_{q+p}(f_0)\\ &\quad + C(N,q+r)\int_0^t (1+(t-s)^{(q+r)/2})\mathbb{N}_{q+r}(M[f](s))ds\\ &\le C(N,q+r)(1+T^{(q+r)/2})\mathbb{N}_{q+r}(f_0)\\ &\quad + C(N,q+r)(1+T^{(q+r)/2})\int_0^t \mathbb{N}_{q+r}(f(s))ds.\end{aligned}$$

Then, the Gronwall's lemma implies that

$$\mathbb{N}_{q+r}(f(t)) \le a\exp(at)\mathbb{N}_{q+r}(f_0), \quad \text{for } t \in [0, T], \tag{3.4}$$

where $a = C(N,q+r)(1+T^{r/2})$. Similarly, we have

$$\begin{aligned}\mathbb{N}_{q,r}(f(t)) &\le C(N,q,r)\max\{1, T^r\}(1+T^{\frac{q}{2}+2r})\\ &\quad \times \left\{[\mathbb{N}_{q,r}(f_0) + T^r\mathbb{N}_{q+r}(f_0)] + \int_0^t [\mathbb{N}_{q,r}(f(s)) + T^r\mathbb{N}_{q+r}(f(s))]ds\right\}\\ &\le C(N,q,r)\max\{1, T^r\}(1+T^{\frac{q}{2}+2r})T^r[\mathbb{N}_{q,r}(f_0) + \exp(aT)\mathbb{N}_{q+r}(f_0)]\\ &\quad + C(N,q,r)\max\{1, T^r\}(1+T^{\frac{q}{2}+2r})\int_0^t \mathbb{N}_{q,r}(f(s))ds.\end{aligned}$$

Then, by the Gronwall's lemma we get

$$\mathbb{N}_{q,r}(f(t)) \le bT^r\exp(bt)[\mathbb{N}_{q,r}(f_0) + \exp(aT)\mathbb{N}_{q+r}(f_0)], \quad \text{for } t \in [0, T], \tag{3.5}$$

where $b = C(N,q,r)\max\{1, T^r\}(1+T^{\frac{q}{2}+2r})$.

Secondly, let $g = M[f] - f$, then $g(t,x,\xi) \ge -f(t,x,\xi)$. Hence, Lemma 2.4 implies that

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,\xi)d\xi \ge \frac{\delta}{1+|x|^r}, \quad \text{for } (t,x) \in [0, T]\times\mathbb{R}^N \text{ a.e.} \tag{3.6}$$

Now, let $K_1(t) = a\exp(at)\mathbb{N}_{q+r}(f_0)$, $K_1(t) = bT^r\exp(bt)[\mathbb{N}_{q,r}(f_0)+\exp(aT)\mathbb{N}_{q+r}(f_0)]$ and

$$\begin{aligned}X_T = \{&f(t,x,\xi) \ge 0\colon\ \mathbb{N}_{q+r}(f(t)) \le K_1(t),\ \mathbb{N}_{q+r}(f(t)) \le K_2(t) \text{ for } t \in [0, T]\\ &\text{and } (1+|x|^r)\rho(t,x) \ge \delta \text{ for } (t,x) \in [0, T]\times\mathbb{R}^N \text{ a.e.}\}.\end{aligned}$$

We define the nonlinear operator F as follows

$$(Ff)(t,x,\xi) = \exp(-t)G(t)f_0(x,\xi) + \int_0^t \exp(-(t-s))G(t-s)M[f](s,x,\xi)ds,$$

then (3.4), (3.5), and (3.6), and their proofs imply that F maps X_T into itself.

On the other hand, it is known [20] that the nonlinear operator $M[f]$ is a Lipschitz mapping from X_T into itself, i.e., there is a positive constant L such that for any $f_1, f_2 \in X_T$

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}(1+|\xi|^2)|M[f_1]-M[f_2]|(t,x,\xi)dxd\xi\\&\quad\le L\int_{\mathbb{R}^N\times\mathbb{R}^N}(1+|\xi|^2)|f_1-f_2|(t,x,\xi)dxd\xi,\quad t\in[0,\ T].\end{aligned}$$

Using the above facts and the Banach fixed point theorem, it is easy to show that there exists a unique nonnegative solution to (1.1)–(1.3) having the desired properties. The proof of Proposition 3.2 is complete. □

Now, we are in a position to construct approximate solutions to (1.1)–(1.3). We always assume that the initial datum f_0 satisfies (1.4). Following the method in [27], we cut off the initial datum f_0 and obtain a sequence of the cutoff initial data as follows

$$f_0^n(x,\xi)=\varphi_n(x,\xi)\max\{f_0(x,\xi),\ n\}+\frac{1}{n}\frac{\exp(-|\xi|^2)}{(1+|x|^r)},$$

where φ_n is any cutoff function such that $0\le\varphi_n\le 1$ and $\varphi_n(x,\xi)=0$ for $|x|^2+|\xi|^2\ge n^2$, and $\lim_{n\to\infty}\varphi(x,\xi)=1$; furthermore, $r\ge N+1$ is sufficiently large. Then,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}(1+|\xi|^\beta)|f_0^n-f_0|(x,\xi)dxd\xi=0,\qquad \lim_{n\to\infty}\|f_0^n-f_0\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}=0.$$

It is also obvious that for $n=1,2,\dots,$ f_0^n as an initial datum satisfies all assumptions of Proposition 3.2. Hence, the approximate Fokker-Planck-BGK equation

$$\frac{\partial}{\partial t}f_n+\xi\cdot\nabla_x f_n-\sigma\Delta_\xi f_n=M[f_n]-f_n,\qquad f_n(0,x,\xi)=f_0^n(x,\xi),\tag{3.7}$$

has a unique global positive solution $f_n(t,x,\xi)$, where

$$M[f_n](t,x,\xi)=\frac{\rho_n(t,x)}{(2\pi\theta_n(t,x))^{N/2}}\exp\left(-\frac{|\xi-u_n(t,x)|^2}{2\theta_n(t,x)}\right)\tag{3.8}$$

and

$$\begin{pmatrix}\rho_n\\ \rho_n u_n\\ \rho_n|u_n|^2+N\rho_n\theta_n\end{pmatrix}(t,x)=\int_{\mathbb{R}^N}\begin{pmatrix}1\\ \xi\\ |\xi|^2\end{pmatrix}f_n(t,x,\xi)d\xi.\tag{3.9}$$

Furthermore, for any $T<\infty$ there is a positive constant $C(T,n,q,r)$ such that for any $t\in[0,\ T]$ and any $q>N+2$

$$\mathbb{N}_{q,r}(f_n(t))\le C(T,n,q,r),\qquad \mathbb{N}_{q+r}(f_n(t))\le C(T,n,q,r).\tag{3.10}$$

Generally speaking, the L^∞ bound $C(T,n,q,r)$ in (3.10) depends not only on T but also on $n,\ q$ and r. In order to obtain the compactness, we need some uniform estimates of the approximate solutions. To this end, we establish the following lemmas.

Lemma 3.3 *Assume that the initial datum f_0 satisfies (1.4), then for $n = 1, 2, \dots$,*

$$\begin{aligned} &\rho_n(t) = \rho_n(0), \quad M_n(t) = M_n(0), \quad t \geq 0, \\ &E_n(t) = E_n(0) + 2N\sigma\rho_n(0)t, \quad t \geq 0, \end{aligned} \tag{3.11}$$

where for $t \geq 0$,

$$(\rho_n, M_n, E_n)(t) = \int_{\mathbb{R}^{2N}} (1, \xi, |\xi|^2) f_n(t, x, \xi) dx d\xi.$$

Furthermore, there exists a positive constant $C = C(N, \beta)$ such that for $t > 0$, $n = 1, 2, \dots$,

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta f_n(t, x, \xi) dx d\xi \leq \exp(Ct) \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta f_0^n dx d\xi + \rho_n(0) \right). \tag{3.12}$$

Proof Equation (3.11) is a direct consequence of the rapidly decreasing of f_n at infinity, conservation of mass and momentum, and linear increasing of kinetic energy. Next, we prove (3.12). Multiplying both sides of (3.7) by $|\xi|^\beta$ and integrating it against x, ξ, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta f_n dx d\xi + \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta (\xi \cdot \nabla_x f_n - \sigma \Delta_\xi f_n) dx d\xi \\ &\quad = \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta (M[f_n] - f_n) dx d\xi \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta M[f_n] dx d\xi. \end{aligned}$$

Thanks to the fast decaying property (3.10) of f_n at infinity and integrating by parts, we obtain,

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta f_n dx d\xi \\ &\quad \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta M[f_n] dx d\xi + \sigma\beta(N + \beta - 2) \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^{\beta-2} f_n dx d\xi \\ &\quad \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta M[f_n] dx d\xi + C_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} |\xi|^\beta f_n dx d\xi + C_1 \rho_n(0), \end{aligned} \tag{3.13}$$

where $C_1 = C_1(N, \sigma, \beta) = \sigma\beta(N + \beta - 2)$.

On the other hand, we have for any nonnegative function $f(\xi)$

$$\begin{aligned} &\int_{\mathbb{R}^N} |\xi|^\beta M[f](\xi) d\xi \\ &\quad = \int_{\mathbb{R}^N} \frac{\rho}{(2\pi\theta)^{N/2}} \exp\left(-\frac{|\xi - u|^2}{2\theta} \right) |\xi|^\beta d\xi \\ &\quad \leq 2^{\beta-1} \int_{\mathbb{R}^N} [|\xi - u|^\beta + |u|^\beta] \frac{\rho}{(2\pi\theta)^{N/2}} \exp\left(-\frac{|\xi - u|^2}{2\theta} \right) d\xi \\ &\quad = 2^{\beta-1} \left(\int_{\mathbb{R}^N} \frac{\rho}{(2\pi\theta)^{N/2}} \exp\left(-\frac{|\xi|^2}{2\theta} \right) |\xi|^\beta d\xi + \rho|u|^\beta \right) \\ &\quad = 2^{\beta-1} \left(\omega_{N-1} \int_0^\infty \frac{\rho r^{N+\beta-1}}{(2\pi\theta)^{N/2}} \exp\left(-\frac{r^2}{2\theta} \right) dr + \rho|u|^\beta \right) \end{aligned}$$

$$= 2^{\beta-1}\left(\frac{2^{\frac{\beta-2}{2}}\omega_{N-1}}{\pi^{N/2}N^{\beta/2}}\Gamma\left(\frac{N+\beta}{2}\right)\rho(N\theta)^{\beta/2}+\rho|u|^{\beta}\right),$$

where ω_{N-1} is the area of the unit sphere of dimension $N-1$. Letting

$$C_2 = C_2(N,\beta) = 2^{\beta-1}\left(2^{\frac{\beta-2}{2}}\omega_{N-1}\pi^{-N/2}N^{-\beta/2}\Gamma\left(\frac{N+\beta}{2}\right)+1\right),$$

we obtain by using the Hölder's inequality,

$$\int_{\mathbb{R}^N}|\xi|^{\beta}M[f](\xi)d\xi \le C_2\rho[|u|^{\beta}+(N\theta)^{\beta/2}]$$
$$\le C_2\rho[|u|^2+(N\theta)]^{\beta/2} = C_2\rho^{\frac{2-\beta}{2}}\left(\int_{\mathbb{R}^N}|\xi|^2 f(\xi)d\xi\right)^{\beta/2}$$
$$\le C_2\int_{\mathbb{R}^N}|\xi|^{\beta}f(\xi)d\xi.$$

Inserting this inequality into the right hand side of (3.13), we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^{\beta}f_n dx d\xi \le C\rho_n(0) + C\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^{\beta}f_n dx d\xi,$$

where $C = C_1 + C_2$. This implies by the Gronwall's lemma that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^{\beta}f_n dx d\xi \le \exp(Ct)\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^{\beta}f_0^n dx d\xi + \rho_n(0)\right), \quad t\ge 0,\ n=1,2,\ldots.$$

The proof is complete. □

Lemma 3.4 *Assume that the initial datum f_0 satisfies* (1.4). *Then, there exist positive constants $C_1 = C_1(N,p)$ and $C_2 = C_2(N,p)$ such that*

$$\|f_n(t)\|_{L^p(\mathbb{R}^{2N})} \le \exp(C_1 t)\left(\|f_0\|_{L^p(\mathbb{R}^{2N})} + \frac{C_2}{n}\right), \quad t>0,\ n=1,2,\ldots. \tag{3.14}$$

Proof It follows from (2.3) that

$$f_n(t,x,\xi) = G(t)f_0^n(x,\xi) + \int_0^t G(t-s)\{M[f_n]-f_n\}(s)ds.$$

Taking L^p norm on both sides, and then using Lemma 2.1(3) and inequality (3.1) in Lemma 3.1, we obtain

$$\|f_n(t)\|_{L^p(\mathbb{R}^{2N})} \le \|f_0^n\|_{L^p(\mathbb{R}^{2N})} + C(N,p)\int_0^t \|f_n(s)\|_{L^p(\mathbb{R}^{2N})}ds, \quad t>0,$$

which implies that

$$\|f_n(t)\|_{L^p(\mathbb{R}^{2N})} \le \exp(C(N,p)t)\|f_0^n\|_{L^p(\mathbb{R}^{2N})}, \quad t>0.$$

On the other hand, by the definition of f_0^n, we get for $1 < p < \infty$

$$\int_{\mathbb{R}^{2N}} [f_0^n]^p dx d\xi \le \int_{\mathbb{R}^{2N}} f_0^p dx d\xi + \frac{\tilde{C}(N,p)}{n^p}.$$

For $p = \infty$, it is obvious that

$$\|f_0^n\|_{L^\infty(\mathbb{R}^{2N})} \le \|f_0\|_{L^\infty(\mathbb{R}^{2N})} + \frac{1}{n}.$$

Letting $C_1 = C(N,p)$ and $C_2 = 1 + \tilde{C}(N,p)$, we obtain the desired result. The proof is complete. □

Remark 3.1 The uniform L^p bound proven in Lemma 3.4 implies that for $1 < p < \infty$, $\{f_n(t,x,\xi)\colon n = 1,2,\dots\}$ is relatively compact in $L^p([0,\ T] \times \mathbb{R}^{2N})$ in the weak topology; and for $p = \infty$, $\{f_n(t,x,\xi)\colon n = 1,2,\dots\}$ is relatively compact in $L^\infty([0,\ T] \times \mathbb{R}^{2N})$ in the weak $\star$ topology [18, 25]. On the other hand, the uniform L^p bound as well as the results of Lemma 3.3 will imply the weak compactness of the local Maxwellians $M[f_n]$ which is important to show the existence of global solutions to the Fokker-Planck-BGK equation. It is also used to establish the L^p bound in Theorem 1.1.

4 Proof of Theorem 1.1

Proof of Theorem 1.1 Firstly, we show that for any $T < \infty$, $\{f_n(t,x,\xi) :\ n = 1,2,\dots\}$ is locally compact in $L^1([0,\ T] \times \mathbb{R}^{2N})$. Let $g_n(t,x,\xi) = M[f_n](t,x,\xi) - f_n(t,x,\xi)$ for $n = 1,2,\dots$, and let $\psi_r(x)$ be the cut-off function corresponding to the ball $B(0,r) \subset \mathbb{R}^N_x$ with radius $r > 0$. We consider the functions $\tilde{f}_n(t,x,\xi) = \psi_r(x) f_n(t,x,\xi)$ and $\tilde{g}_n(t,x,\xi) = \psi_r(x) g_n(t,x,\xi) - [\xi \cdot \nabla_x \psi_r(x)] f_n(t,x,\xi)$, it follows from (3.7) that they satisfy (2.4), i.e.,

$$\frac{\partial}{\partial t}\tilde{f}_n + \mathcal{L}\tilde{f}_n = \tilde{g}_n(t,x,\xi), \qquad \tilde{f}_n(0,x,\xi) = \tilde{f}_0^n(x,\xi),$$

where $\tilde{f}_0^n(x,\xi) = \psi_r(x) f_0^n(x,\xi)$. For $R > 0$ large enough, we have by (3.11) of Lemma 3.3,

$$\begin{aligned}
&\int_0^T dt \int_{|x|^2+|\xi|^2>R^2} |\psi_r(x) g_n(t,x,\xi)| dx d\xi \\
&\le \int_0^T dt \int_{|x|^2+|\xi|^2>R^2} \psi_r(x)\{M[f_n] + f_n\}(t,x,\xi) dx d\xi \\
&\le \int_0^T dt \int_{\{|\xi|^2>R^2-(r+1)^2,\ |x|\le r+1\}} \{M[f_n] + f_n\}(t,x,\xi) dx d\xi \\
&\le \int_0^T dt \int_{|\xi|^2>R^2-(r+1)^2} \{M[f_n] + f_n\}(t,x,\xi) dx d\xi \\
&\le \frac{2}{R^2-(r+1)^2} \int_0^T dt \int_{\mathbb{R}^{2N}} |\xi|^2 f_n(t,x,\xi) dx d\xi
\end{aligned}$$

$$= \frac{2}{R^2-(r+1)^2}\int_0^T dt \int_{\mathbb{R}^{2N}} (|\xi|^2+2N\sigma t) f_0^n(x,\xi)dxd\xi$$
$$\le \frac{2}{R^2-(r+1)^2}\int_0^T dt \left(\int_{\mathbb{R}^{2N}} (|\xi|^2+2N\sigma t) f_0(x,\xi)dxd\xi + C\right).$$

This implies that

$$\lim_{R\to\infty}\sup_{n\ge 1}\int_0^T dt \int_{|x|^2+|\xi|^2>R^2} |\psi_r(x) g_n(t,x,\xi)|dxd\xi = 0.$$

Since $|\xi\cdot\nabla_x\psi_r(x)| \le C_1\psi_r(x)|\xi|$ for some positive constant C_1, a similar argument gives that

$$\lim_{R\to\infty}\sup_{n\ge 1}\int_0^T dt \int_{|x|^2+|\xi|^2>R^2} |[\xi\cdot\nabla_x\psi_r(x)] f_n(t,x,\xi)|dxd\xi = 0.$$

The above two equalities obviously imply that

$$\lim_{R\to\infty}\sup_{n\ge 1}\int_0^T dt \int_{|x|^2+|\xi|^2>R^2} |\tilde{g}_n(t,x,\xi)|dxd\xi = 0. \tag{4.1}$$

On the other hand, it is easy to show

$$\lim_{R\to\infty}\sup_{n\ge 1}\int_0^T dt \int_{|x|^2+|\xi|^2>R^2} |\tilde{f}_0^n(x,\xi)|dxd\xi = 0. \tag{4.2}$$

It follows from (4.1), (4.2) and Lemma 2.2 that $\{\tilde{f}_n(t,x,\xi)\colon n=1,2,\dots\}$ is relatively compact in $L^1([0,\ T]\times\mathbb{R}^{2N})$. This implies that for any $r>0$, $\{f_n(t,x,\xi)\colon n=1,2,\dots\}$ is relatively compact in $L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N)$. Hence, without loss of generality, we may assume that $\{f_n(t,x,\xi)\colon n=1,2,\dots\}$ converges to a nonnegative function $f(t,x,\xi)\in L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N)$ in the strong topology for any $r>0$. Using elementary cut off method, we also obtain by passing to the limit in (3.12),

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} |\xi|^\beta f(t,x,\xi)dxd\xi \le \exp(Ct)\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}|\xi|^\beta f_0 dxd\xi + \rho(0)\right),\quad t>0, \tag{4.3}$$

which obviously implies (1.6).

Secondly, we show the global existence of a nonnegative solution to (1.1)–(1.3). We will see that the solution is the limit $f(t,x,\xi)$ of the approximate solutions $\{f_n(t,x,\xi)\colon n=1,2,\dots\}$. We begin by establishing that for any $r>0$,

$$|\xi|^2 f_n(t,x,\xi)\to|\xi|^2 f(t,x,\xi),\quad \text{in } L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N_\xi) \tag{4.4}$$

as $n\to\infty$. Actually, for any $R>0$,

$$\int_0^T dt\int_{B(0,r)} dx\int_{\mathbb{R}^N}|\xi|^2|f_n(t,x,\xi)-f(t,x,\xi)|d\xi$$
$$\le R^2\int_0^T dt\int_{B(0,r)} dx\int_{|\xi|\le R}|f_n(t,x,\xi)-f(t,x,\xi)|d\xi$$

$$+\frac{1}{R^{\beta-2}}\int_0^T dt\int_{B(0,r)}dx\int_{|\xi|>R}|\xi|^\beta|f_n(t,x,\xi)-f(t,x,\xi)|d\xi$$
$$\le R^2\int_0^T dt\int_{B(0,r)}dx\int_{\mathbb{R}^N}|f_n(t,x,\xi)-f(t,x,\xi)|d\xi$$
$$+\frac{1}{R^{\beta-2}}\int_0^T dt\int_{\mathbb{R}^{2N}}|\xi|^\beta[|f_n(t,x,\xi)|+|f(t,x,\xi)|]dxd\xi =: I_1+I_2.$$

It follows from (3.12) and (4.3) that there exists a positive constant $C=C(N,\sigma,\beta,T)$ such that $I_2\le C/R^{\beta-2}$. For any $\varepsilon>0$, we can take R large enough to ensure that $I_2<\varepsilon/2$. For such R fixed, we have $\lim_{n\to\infty}\int_0^T dt\int_{B(0,r)}dx\int_{\mathbb{R}^N}|f_n(t,x,\xi)-f(t,x,\xi)|d\xi=0$, due to the convergence of $f_n(t,x,\xi)$ to $f(t,x,\xi)$ in the space $L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N_\xi)$. This observation obviously implies (4.4). It follows from (4.4) that

$$\lim_{n\to\infty}(\rho_n,\ \rho_n u_n,\ \rho_n|u_n|^2+N\rho_n\theta_n)(t,x)=(\rho,\ \rho u,\ \rho|u|^2+N\rho\theta)(t,x),$$
$$\text{in } L^1([0,\ T]\times B(0,r)).$$

Hence, we have for almost all $(t,x)\in[0,\ T]\times\mathbb{R}^N$,

$$\lim_{n\to\infty}(\rho_n,\ \rho_n u_n,\ \rho_n|u_n|^2+N\rho_n\theta_n)(t,x)=(\rho,\ \rho u,\ \rho|u|^2+N\rho\theta)(t,x). \tag{4.5}$$

On the other hand, by Lemmas 3.1, 3.4, and 3.3, we get for $t>0$, $n=1,2,\dots$

$$\|M[f_n](t)\|_{L^p(\mathbb{R}^{2N})}\le C(N,p)\exp(C_1t)(\|f_0\|_{L^p(\mathbb{R}^{2N})}+C_2),$$

and

$$\|(1+|\xi|^2)M[f_n](t)\|_{L^1(\mathbb{R}^{2N})}\le\|(1+|\xi|^2)f_0^n\|_{L^1(\mathbb{R}^{2N})}+2N\sigma t\|f_0^n\|_{L^1(\mathbb{R}^{2N})}.$$

The last two inequalities and Dunford-Pettis theorem [18] imply that there exists a nonnegative function $\omega(t,x,\xi)$ such that (choosing a subsequence if necessary)

$$M[f_n](t,x,\xi)\to\omega(t,x,\xi),\quad \text{weakly in } L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N_\xi). \tag{4.6}$$

By (4.5), (4.6) and the standard procedure developed in [21], it is easy to show that $\omega(t,x,\xi)=M[f](t,x,\xi)$ and $M[f_n](t,x,\xi)\to M[f](t,x,\xi)$ in $L^1([0,\ T]\times B(0,r)\times\mathbb{R}^N_\xi)$ in the strong topology.

Now, by passing to the limit in the approximate Fokker-Planck-BGK equation (3.7)–(3.9), we know that $f(t,x,\xi)$ is a solution to (1.1)–(1.3) in distributional sense.

Thirdly, we show the smoothness of the solution $f(t,x,\xi)$. By Lemma 2.1(2), we know that the operator $G(t)$ maps any $L^q(\mathbb{R}^{2N})$ into $C^\infty((0,\ \infty)\times\mathbb{R}^{2N})\cap C([0,\ \infty):L^q(\mathbb{R}^{2N}))$ (see, for example [8]). On the other hand, the solution satisfies

$$f(t,x,\xi)=G(t)f_0(x,\xi)+\int_0^t G(t-s)\{M[f]-f\}(s,x,\xi)ds.$$

Since $f_0(x,\xi)\in L^1(\mathbb{R}^{2N})$ and $\{M[f]-f\}(s,x,\xi)\in L^\infty((0,\ \infty);\ L^1(\mathbb{R}^{2N}))$, consequently, $f(t,x,\xi)\in C([0,\ \infty):\ L^q(\mathbb{R}^{2N}))\cap C^\infty((0,\ \infty)\times\mathbb{R}^{2N})$.

Finally, we show that the solution $f(t,x,\xi)$ satisfies (1.5), (1.6) and (1.7). Equation (1.6) is already proved in (4.3); to prove (1.5), we multiply both sides of (1.2) by collision invariants 1, ξ and $|\xi|^2$, respectively, and integrate by parts, then the desired results are obtained. To prove (1.7), we use Lemma 3.4 and Remark 3.1. Actually, (1.7) follows from (3.4) and the weak (weak $\star$) convergence of $f_n(t,x,\xi)$ to $f(t,x,\xi)$ in the space $L^p([0,\ T]\times\mathbb{R}^{2N})$ (see, for example [27] for details). The proof of Theorem 1.1 is complete. □

References

1. Arkeryd, L., Maslova, N.: On diffuse reflection at the boundary for the Boltzmann equation and related equations. J. Stat. Phys. **77**, 1051–1077 (1994)
2. Bellouquid, A.: Existence globale et comportement asymptotique du problème de Cauchy pour le modèle de BGK. C. R. Acad. Sci. Ser. I: Math. **321**, 1637–1640 (1995)
3. Bellouquid, A.: Global existence and large-times behavior for BGK model for a gas with non constant cross section. Transp. Theory Stat. Phys. **32**, 157–184 (2003)
4. Bellouquid, A.: Limite asymptotique pour le modèle de BGK. C. R. Acad. Sci. Ser. I: Math. **324**, 951–956 (1997)
5. Bellouquid, A.: On the asymptotic analysis of kinetic models towards the compressible Euler and acoustic equations. Math. Models Methods Appl. Sci. **14**, 853–882 (2004)
6. Bhatnagar, P.L., Gross, E.P., Krook, M.: A model for collision processes in gases. Phys. Rev. **94**, 511–514 (1954)
7. Bouchut, F.: Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions. J. Funct. Anal. **111**, 239–258 (1993)
8. Bouchut, F.: Smoothing effect for the non-linear Vlasov-Poisson-Fokker-Planck system. J. Differ. Equ. **122**, 225–238 (1995)
9. Carrillo, J.A., Soler, J.: On the initial value problem for the Vlasov-Poisson-Fokker-Planck system with initial data in L^p spaces. Math. Methods Appl. Sci. **18**, 825–839 (1995)
10. Carrillo, J.A., Soler, J.: On the Vlasov-Poisson-Fokker-Planck equations with measures in Morrey spaces as initial data. J. Math. Anal. Appl. **207**, 475–495 (1997)
11. Carrillo, J.A., Soler, J., Vázquez, J.L.: Asymptotic behaviour and self-similarity for the three dimensional Vlasov-Poisson-Fokker-Planck system. J. Funct. Anal. **141**, 99–132 (1996)
12. Castella, F.: The Vlasov-Poisson-Fokker-Planck system with infinite kinetic energy. Indiana Univ. Math. J. **47**, 939–963 (1998)
13. Cercignani, C.: Mathematical Methods in Kinetic Theory. Plenum, New York (1990)
14. Desvillettes, L.: Convergence to equilibrium in large time for Boltzmann and B.G.K. equation. Arch. Ration. Mech. Anal. **110**, 73–91 (1990)
15. Diperna, R.J., Lions, P.L.: On the Fokker-Planck-Boltzmann equations. Commun. Math. Phys. **120**, 1–23 (1988)
16. Diperna, R.J., Lions, P.L.: On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann. Math. **130**, 321–366 (1989)
17. Diperna, R.J., Lions, P.L.: Global solutions of Boltzmann's equation and the entropy inequality. Arch. Ration. Mech. Anal. **114**, 47–55 (1991)
18. Dunford, N., Schwartz, J.T.: Linear Operators, I. General Theory. Interscience, New York (1958)
19. Greenberg, W., Polewczak, J.: A global existence theorem for the nonlinear BGK equation. J. Stat. Phys. **55**, 1313–1321 (1989)
20. Mischler, S.: Uniqueness for the BGK-equation in $\mathbb{R}^N$ and rate of convergence for a semi-discrete scheme. Diff. Integral Equ. **9**, 1119–1138 (1996)
21. Perthame, B.: Global existence to the BGK model of Boltzmann equation. J. Differ. Equ. **82**, 191–205 (1989)
22. Perthame, B., Pulvirenti, M.: Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model. Arch. Ration. Mech. Anal. **125**, 289–295 (1993)
23. Rein, G., Weckler, J.: Generic global solutions of the Vlasov-Fokker-Planck-Poisson system in three dimensions. J. Differ. Equ. **99**, 59–77 (1992)
24. Victory, H.D., O'Dwyer, B.P.: On classical solutions of Vlasov-Poisson-Fokker-Planck systems. Indiana Univ. Math. J. **39**, 105–157 (1990)
25. Yosida, K.: Functional Analysis, 6th edn. Springer, New York (1980)
26. Zhang, X., Hu, S.: Moment propagation of the BGK equation. Math. Appl. **19**, 857–862 (2006)
27. Zhang, X., Hu, S.: L^p solutions to the Cauchy problem of the BGK equation. J. Math. Phys. **48**, 113304 (2007)